Math 55 Quiz 4 DIS 105

Name: ______________________________ 28 Feb 2022

1. (a) Use mathematical induction to show that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n. [5 points]

(b) Show that the sum of any 5 consecutive perfect squares is divisible by 5. [5 points]

(a) Let $P(n)$ be the proposition that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
$1^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6}$ so $P(1)$ is true.
Suppose $P(k)$ is true for some positive integer k. Then

$$\begin{aligned}
1^2 + 2^2 + \cdots + (k+1)^2 =& (1^2 + 2^2 + \cdots + k^2) + (k+1)^2 \\
=& \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
=& (k+1)(\frac{k(2k+1)}{6} + k + 1) \\
=& (k+1)\frac{2k^2 + 7k + 6}{6} \\
=& \frac{(k+1)(k+2)(2k+3)}{6} \\
=& \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

so $P(k+1)$ is true.
By mathematical induction, $P(n)$ is true for any positive integer n.

(b) Solution 1: Mathematical induction
Let $Q(n)$ be the proposition that

$$(n+1)^2 + \cdots + (n+5)^2$$

is divisible by 5.
$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$ is divisible by 5, so $Q(0)$ is true. Suppose $Q(k)$ is true for some nonnegative integer k. Then

$$\begin{aligned}
(k+2)^2 + \cdots + (k+6)^2 =& ((k+1)^2 + \cdots + (k+5)^2) + (k+6)^2 - (k+1)^2 \\
=& ((k+1)^2 + \cdots + (k+5)^2) + 5k + 35
\end{aligned}$$

$(k+1)^2+\cdots+(k+5)^2$ and $5k+35$ are divisible by 5, so $(k+2)^2+\cdots+(k+6)^2$ is divisible by 5. In other words, $Q(k+1)$ is true.
By mathematical induction, $Q(n)$ is true for any nonnegative integer n.
Solution 2: Use part (a) and Euclid's lemma

$$\begin{aligned}(n+1)^2+\cdots+(n+5)^2 &= (1^2+2^2+\cdots+(n+5)^2)-(1^2+2^2+\cdots+n^2)\\ &= \frac{(n+5)(n+6)(2n+11)}{6}-\frac{n(n+1)(2n+1)}{6}\\ &= \frac{(n+5)(n+6)(2n+11)-n(n+1)(2n+1)}{6}\end{aligned}$$

Now $(n+5)(n+6)(2n+11)\equiv n(n+1)(2n+1) \pmod 5$ so $(n+5)(n+6)(2n+11)-n(n+1)(2n+1)$ is divisible by 5. Since 5 and 6 are relatively prime, by Euclid's lemma, $(n+1)^2+\cdots+(n+5)^2=\frac{(n+5)(n+6)(2n+11)-n(n+1)(2n+1)}{6}$ is divisible by 5 as well.